Section 14.7

Optimization in Several Variables

PreLecture Video

1 The 2nd Derivative Test

Local and Absolute Extrema

Let $f(x, y)$ be a function of two variables, with domain D.
A point (a, b) in D is...

- a **local maximum** if $f(x, y) \leq f(a, b)$ for (x, y) near (a, b);
- a **local minimum** if $f(x, y) \geq f(a, b)$ for (x, y) near (a, b);
- an **absolute maximum** if $f(x, y) \leq f(a, b)$ for all (x, y) in D;
- an **absolute minimum** if $f(x, y) \geq f(a, b)$ for all (x, y) in D.

Some terminology:

— "extremum" (plural: "extrema") means "minimum or maximum"

— "Global" means the same thing as "absolute"

Critical Points

Fermat's Theorem

Suppose that $f(x, y)$ is differentiable and has a local extremum at (a, b).

Then $f_x(a, b) = f_y(a, b) = 0$. Equivalently, $\nabla f(a, b) = \vec{0}$.

Proof: Suppose $f(a, b)$ is a local maximum. Then $g(x) = f(x, b)$ has a local maximum at $x = a$. By Fermat's Theorem for 1-variable functions, $g'(a) = f_x(a, b) = 0$; similarly $f_y(a, b) = 0$.

Definition

If $\nabla f(a, b) = \vec{0}$, then the point (a, b) is called a **critical point** of f.

- All local extrema are critical points, but not all critical points are necessarily local extrema.
- As in Calculus I, we need a **test** to classify critical points as local maxima, local minima, or neither.

The Second Derivative Test

Let $f(x, y)$ be a function of two variables. The **discriminant** of f at a point (a, b) in the domain is

$$D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2 .$$

Second Derivative Test

If (a, b) is a critical point of f and all second partials f_{xx}, f_{xy}, f_{yy} are continuous near (a, b), then

(I) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a **local minimum.**

(II) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a **local maximum.**

(III) If $D(a, b) < 0$, then (a, b) is a **saddle point.**

(IV) If $D(a, b) = 0$, then the test is **inconclusive.**

▸ Video

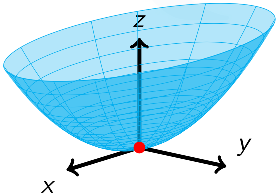
The Second Derivative Test

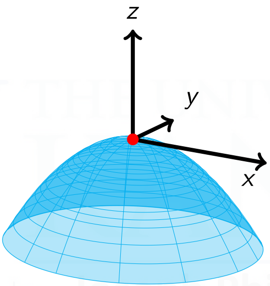
$z = x^2 + 4y^2$
CP: $(0, 0)$

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 8 \end{vmatrix} = 16$$

$D > 0,\ f_{xx} > 0$

local minimum





$z = -2x^2 + xy - 3y^2$
CP: $(0, 0)$

$$D = \begin{vmatrix} -4 & 1 \\ 1 & -6 \end{vmatrix} = 23$$

$D > 0,\ f_{xx} < 0$

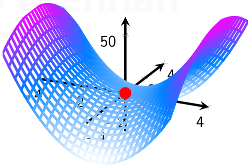
local maximum

$z = 4x^2 + xy - 2y^2$
CP: $(0, 0)$

$$D = \begin{vmatrix} 8 & 1 \\ 1 & -4 \end{vmatrix} = -33$$

$D < 0,\ f_{xx} > 0$

saddle point

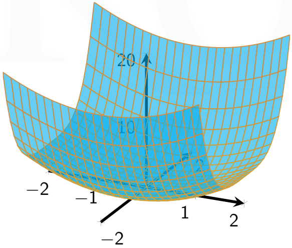


The Second Derivative Test: The Case $D = 0$

$z = x^2 + y^4$
CP: $(0, 0)$

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

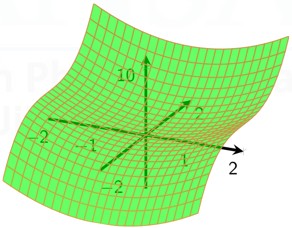
local minimum



$z = x^2 + y^3$
CP: $(0, 0)$

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

not a local extremum



Why The Test Works (Optional)

The second derivative test uses 2^{nd} degree Taylor polynomial approximation of the graph of the function at it's critical points to predict the shape of the graph. ▸ Link

For simplicity, consider the polynomial $f(x, y) = Ax^2 + Bxy + Cy^2$, which has a critical point at $(0, 0)$. ▸ Link

- If $D = 4AC - B^2 > 0$, then the graph of f is an elliptic paraboloid, opening up (if $A, C > 0$) or down (if $A, C < 0$). Hence $(0, 0)$ is a local extremum (min or max respectively). Note that A, C must have the same sign.
- If $D = 4AC - B^2 < 0$ (for example, if A and C have opposite signs) then the graph is a hyperbolic paraboloid, a.k.a. a saddle surface (or "Pringle"). Hence $(0, 0)$ is a saddle point and not a local extremum.
- If $D = 4AC - B^2 = 0$, then $f(x, y)$ factors as a perfect square, and the graph is a cylinder over a parabola. Technically $(0, 0)$ is a local extremum, but $f(x, y)$ has the same value along an entire line containing $(0, 0)$.

Why The Test Works (Optional)

- The discriminant test uses the *quadratic approximation* $Q(x, y)$ of $f(x, y)$ — the quadric surface that fits its graph most closely.
- In fact, $Q(x, y)$ is a multivariable Taylor polynomial of degree 2, with the same first and second partial derivatives as f, and therefore the same discriminant.
- If $D \neq 0$, then the third- and higher-order terms are insignificant and the critical point has the same behavior relative to Q as it does to f.
- If $D = 0$, then the test is inconclusive — you need to look at higher-order terms (or do something else).

Sketch of the Proof (Video)

Example 1: Find and classify the critical points of

$$f(x,y) = 3x^2 - 6xy + 5y^2 + y^3$$

Solution: The critical points are those that satisfy $\nabla f(x,y) = \vec{0}$.

$$\nabla f(x,y) = \left\langle 6x - 6y,\ -6x + 10y + 3y^2 \right\rangle$$

$$\begin{cases} 6x - 6y = 0 \\ -6x + 10y + 3y^2 = 0 \end{cases} \quad \Longrightarrow \quad (x,y) = (0,0) \text{ or } \left(-\frac{4}{3}, -\frac{4}{3}\right)$$

Now use the Second Derivative Test to classify the critical points:

$$D(x,y) = f_{xx} f_{yy} - [f_{xy}]^2 = 36y + 24$$

Critical point	**Second Deriv. Test** $D(a,b)$	$f_{xx}(a,b)$	**Classification**
$(0,0)$	24	6	Local minimum
$\left(-\frac{4}{3}, -\frac{4}{3}\right)$	-24		Saddle point

Video

Optimization in Three Variables (Optional)

How do we find local extrema of a function $f(x, y, z)$ of three variables?

1. **Find critical points.** They are the solutions of the equation

$$\nabla f(a,b,c) = 0 \qquad \text{or equivalently} \qquad \begin{cases} f_x(a,b,c) = 0 \\ f_y(a,b,c) = 0 \\ f_z(a,b,c) = 0 \end{cases}$$

2. **Classify them.** Now we need three discriminants:

$$D_1 = f_{xx}(a,b,c) \qquad D_2 = \begin{vmatrix} f_{xx}(a,b,c) & f_{xy}(a,b,c) \\ f_{yx}(a,b,c) & f_{yy}(a,b,c) \end{vmatrix}$$

$$D_3 = \begin{vmatrix} f_{xx}(a,b,c) & f_{xy}(a,b,c) & f_{xz}(a,b,c) \\ f_{yx}(a,b,c) & f_{yy}(a,b,c) & f_{yz}(a,b,c) \\ f_{zx}(a,b,c) & f_{zy}(a,b,c) & f_{zz}(a,b,c) \end{vmatrix}$$

Optimization in Three or More Variables (Optional)

Second Derivative Test — Three Variables

If $P(a, b, c)$ is a critical point of f and all second partials are continuous near P, then

(I) If $\boldsymbol{D_1} > 0$, $\boldsymbol{D_2} > 0$, and $\boldsymbol{D_3} > 0$, then P is a **local minimum.**

(II) If $\boldsymbol{D_1} < 0$, $\boldsymbol{D_2} > 0$, and $\boldsymbol{D_3} < 0$, then P is a **local maximum.**

(III) If $\boldsymbol{D_3} \neq 0$ but neither (I) nor (II) occurs, then P is **not** a local extrema.

(IV) If $\boldsymbol{D_3} = 0$, then the test is **inconclusive.**

- To classify critical points of functions of n variables (Optional)
  Use discriminants $D_1, \ldots, D_k, \ldots, D_n$, which are $k \times k$ determinants
  (1) All discriminants positive $\implies$ local minimum
  (2) Alternating sign pattern starting with $D_1 < 0$ $\implies$ local maximum

Optimization in Three Variables (Optional)

Example (Optional): Find and classify the critical points of the function

$$f(x, y, z) = x^3 + x^2 + y^2 + z^2 + 5z.$$

Solution: The critical points are the solutions of $\nabla f(x, y, z) = \vec{0}$, i.e.,

$$\begin{cases} 3x^2 + 2x = 0 \\ 2y = 0 \\ 2z + 5 = 0 \end{cases} \implies \begin{array}{l} P(0, 0, -5/2) \\ Q(-2/3, 0, -5/2) \end{array}$$

Matrix of second-order partials ("Hessian") and discriminants:

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 6x+2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{array}{l} D_1 = 6x + 2 \\ D_2 = 12x + 4 \\ D_3 = 24x + 8 \end{array}$$

Optimization in Three Variables (Optional)

Example (continued):

$$D_1 = 6x + 2$$
$$D_2 = 12x + 4$$
$$D_3 = 24x + 8$$

Critical point	D_1	D_2	D_3	Sign pattern	Classification
$P(0,\ 0,\ -5/2)$	2	4	8	$+++$	Local minimum
$Q(-2/3,\ 0,\ -5/2)$	-2	-4	-8	$---$	Not a local extremum

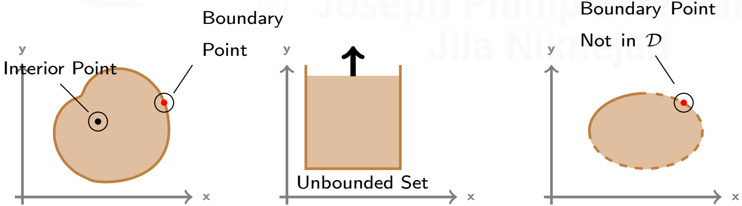
2 Absolute Exrema

The Extreme Value Theorem

Extreme Value Theorem

If $z = f(x, y)$ is continuous on a **closed** and **bounded** set D in $\mathbb{R}^2$, then $f(x, y)$ attains an absolute maximum and an absolute minimum.

- "Closed" means that D contains all the points on its boundary.
- "Bounded" means that D does not go off to infinity in some direction.
  (Disks are bounded; so is any set contained in *some* disk.)



The Closed/Bounded Domain Method

Extreme Value Theorem

If $z = f(x, y)$ is continuous on a **closed** and **bounded** set D in $\mathbb{R}^2$, then $f(x, y)$ attains an absolute maximum and an absolute minimum.

Closed/Bounded Domain Method to find absolute extrema:

(I) Find all critical points.

(II) Find the extrema of f on the **boundary** of D.

(III) The points found from (I) and (II) with the largest/smallest value(s) of f are the absolute extrema.

The Second Derivative test isn't required.
However, step (II) can be very complicated!

The Closed/Bounded Domain Method

Example 2: Find the absolute extrema of $f(x,y) = x^2 - 4xy + y^2$ on $D = \{(x,y)\,|\,x^2+y^2 \leq 1\}$.

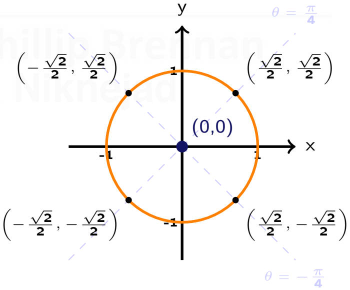
Solution: **(I)** Find the critical points (a,b) in D:

$$\nabla f(x,y) = \langle 2x - 4y,\ -4x + 2y \rangle$$

$(0,0)$ is the only critical point.

(II) The boundary of D is the unit circle $x^2+y^2 = 1$, which can be parametrized $x = \cos(\theta)$, $y = \sin(\theta)$, $0 \leq \theta \leq 2\pi$.

$$g(\theta) = f(\cos(\theta), \sin(\theta)) = 1 - 2\sin(2\theta)$$
$$g'(\theta) = -4\cos(2\theta) = 0$$
$$\theta = k\pi/4 \quad (k \text{ odd})$$



The Closed/Bounded Domain Method

Example 2 (cont'd): Find the absolute extrema of $f(x, y) = x^2 - 4xy + y^2$ on $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Solution: **(III)** Find the values of f at all critical points. Link

Critical point	Value of f	Classification
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	-1	absolute minimum
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	3	absolute maximum
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	3	absolute maximum
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	-1	absolute minimum
$(0, 0)$	0	not an extremum